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CLOSURE TECHNIQUES FOR INFINITE SYSTEMS OF DIFFERENTIAL EQUATIONS

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The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

Part of the RAND research program consists of basic supporting studies in mathematics. The mathematical research in this Memorandum is concerned with techniques for obtaining approximate solutions to systems of nonlinear differential equations, which arise in a variety of physical problems.

SUMMARY

In some recent papers, [1], [2], [3], we have applied various closure techniques to the problem of obtaining approximate solutions of nonlinear differential equations. These questions fall within the area of differential approximation. Starting with a vector system

$$\frac{dx}{dt} = g(x),$$

we replace it by another system

$$\frac{dy}{dt} = h(y),$$

with simpler analytic and computation properties in such a way that $\| x - y \|$ is small. In some cases we are content merely to have $h(y)$ linear, i.e., of the form $Ay + b$; in other cases, we want the dimension of y to be considerably less than that of x.

In this paper we wish to consider the case where the original system is infinite dimensional and the approximating system is to be finite dimensional. Infinite dimensional systems arise in a natural fashion from the consideration of partial differential equations.

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CLOSURE TECHNIQUES FOR INFINITE SYSTEMS OF DIFFERENTIAL EQUATIONS

1. INTRODUCTION

In some recent papers, [1], [2], [3], we have applied various closure techniques to the problem of obtaining approximate solutions of nonlinear differential equations. These questions fall within the area of differential approximation. Starting with a vector system

$$(1.1) \qquad \frac{dx}{dt} = g(x),$$

we replace it by another system

$$(1.2) \qquad \frac{dy}{dt} = h(y),$$

with simpler analytic and computation properties, in such a way that $\| x - y \|$ is small. In some cases we are content merely to have $h(y)$ linear, i.e., of the form $Ay + b$; in other cases, we want the dimension of y to be considerably less than that of x.

In this paper we wish to consider the case where the original system is infinite dimensional and the approximating system is to be finite dimensional. Infinite dimensional systems arise in a natural fashion from the consideration of partial differential equations. For example, if we look for a solution of

$$(1.3) \qquad u_t + u\,u_x = \epsilon\, u_{xx}$$

in the form

(1.4) $$u(x, t) = \sum_k u_k(t)\, e^{ikx},$$

we obtain an infinite system of ordinary differential equations for the components $u_k(t)$; see [4].

The problem of closure is now more difficult since there is no immediate way of expressing the components corresponding to higher harmonies in terms of the components for smaller k. In this paper we shall discuss two approaches.

2. EXTRAPOLATION AND SELF-CONSISTENCY

Let us suppose that we have an infinite linear system of the form

(2.1) $$\frac{du_k}{dt} = \sum_{\ell=1}^{\infty} a_{k\ell}\, u_\ell\,, \quad u_k(o) = c_k\,, \quad k = 1, 2, \ldots$$

For a discussion of existence and uniqueness of solutions to equations of this type, and an examination of when the sections

(2.2) $$\frac{du_k}{dt} = \sum_{\ell=1}^{N} a_{k\ell}\, u_\ell\,, \quad u_k(o) = c_k, \ k = 1, 2, \ldots, N,$$

yield solutions which converge to the solutions of (2.1), see [5].

Instead of using the closure technique implicit in (2.2), namely, $u_k = 0$, $k \geq N + 1$, we wish to approximate to the remainder by means of linear combinations of the preceding u_k,

$$(2.3) \qquad \sum_{\ell \geq N+1} a_{k\ell} = \sum_{\ell=1}^{N} b_{k\ell} u_\ell ,$$

see [1], [2], [3].

The coefficients $b_{k\ell}$ are to be chosen so that

$$(2.4) \qquad \int_0^T \Big[\sum_{\ell \geq N+1} a_{k\ell} u_\ell - \sum_{\ell=1}^{N} b_{k\ell} u_\ell \Big]^2 dt$$

is a minimum. We obtain in this way the linear algebraic equations for the $b_{k\ell}$

$$(2.5) \qquad \sum_{\ell \geq N+1} a_{k\ell} \int_0^T u_\ell u_r \, dt = \sum_{\ell=1}^{N} b_{r\ell} \int_0^T u_\ell u_r \, dt ,$$

$$r = 1, 2, \ldots, N.$$

The usual difficulty now confronts us. How do we compute the integrals $\int_0^T u_\ell u_r \, dt$ involving the unknown solution?

Consider first the case where $1 \leq r, \ell \leq N$. To obtain these integrals, we use the finite system in (2.2). Call the solutions $u_r^{(o)}$. The coefficients of $b_{r\ell}$ in (2.5) are then $\int_0^T u_\ell^{(o)} u_r^{(o)} dt$. Observe that these quantities can be computed directly in the course of obtaining the $u_k^{(o)}$ by adjoining to (2.2) the equations

$$(2.6) \qquad \frac{dw_{r\ell}}{dt} = u_r u_\ell , \quad w_{r\ell}(o) = 0 ,$$

and asking only for the values $w_{r\ell}(T)$.

The more difficult problem is that of the calculation of $\int_0^T u_r u_\ell \, dt$ for $r = 1, 2, \ldots, w$, $\ell \geq N + 1$. Here we use extrapolation techniques. Keep r fixed and let ℓ vary over the integers, $\ell = 1, 2, \ldots$ It is reasonable to suspect that $f_{\ell,r} = \int_0^T u_r u_\ell \, dt$ will be a well-behaved sequence. Consequently, if we possess the values $f_{1,r}, f_{2,r}, \ldots, f_{N,r}$, we can use any of a number of extrapolation techniques (see [6]) to obtain the values of $f_{\ell,r}$ for $\ell \geq N + 1$.

The system in (2.5) then has the form

$$\text{(2.7)} \qquad \sum_{M \geq \ell \geq N+1} a_{k\ell} f_{\ell,r} = \sum_{\ell=1}^{N} b_{r\ell} w_{r\ell}(T) .$$

Here M is a cut-off number, such as 2N, which depends upon the size of the coefficients $a_{k\ell}$ and the rapidity of convergence of the infinite series.

Solving (2.7) numerically, we obtain the coefficients $b_{r\ell}^{(o)}$. We use the superscript to indicate the fact that these are the first approximations. To obtain higher approximations, we use self-consistency techniques.

In place of (2.2), let us now use

$$\text{(2.8)} \qquad \frac{du_k}{dt} = \sum_{\ell=1}^{N} a_{k\ell} u_\ell + \sum_{\ell=1}^{N} b_{k\ell}^{(o)} u_\ell , \quad u_k(o) = c_k .$$

Call the solutions of this equation $u_k^{(1)}$, $k = 1, 2, \ldots, N$. We now proceed as before to calculate $w_{r\ell}^{(1)}(T)$, $f_{\ell,r}^{(1)}$ and

$b_{k\ell}^{(1)}$. With the new coefficients $b_{k\ell}^{(1)}$, we introduce the equation

$$(2.9) \qquad \frac{du_k}{dt} = \sum_{\ell=1}^{N} a_{k\ell}\, u_\ell + \sum_{\ell=1}^{N} b_{k\ell}^{(1)}\, u_\ell, \quad u_k(o) = c_k.$$

This process is repeated until the values of the $b_{k\ell}$ settle down.

3. NONLINEAR EQUATIONS

In the case where the infinite system arises from a nonlinear equation such as (1.3), we can obtain a finite system in a simple fashion. Write

$$(3.1) \qquad u_N = \sum_{|k| \leq N} u_k(t) e^{ikx} ,$$

replace $u\, u_x$ by $u_N (u_N)_x$, and in this fashion obtain the higher harmonics in terms of the lower harmonics.

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